Fair division

Participants:

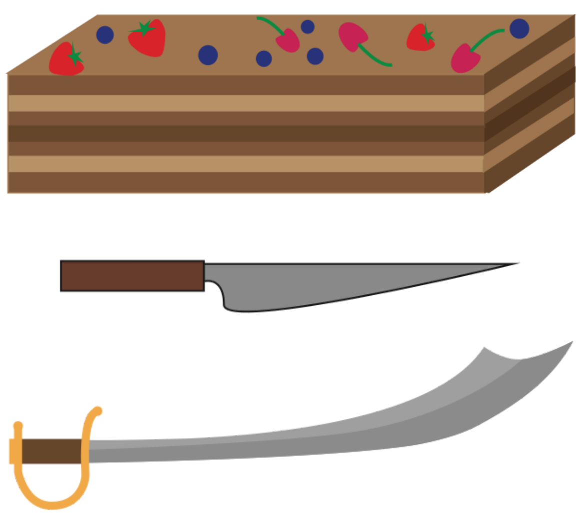
Ages 12 and up, divided into groups of 4-5 people.

Overview:

We explore different mathematical ways of dividing a cake between several people, so they all get a fair share. In our final activity, we split chores instead of cake. Although everyone wants the least amount of chores (instead of the largest portion), we can divide them fairly with a similar method.

Preparations

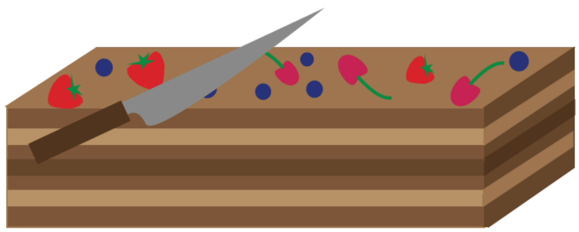
Prepare cardboard knives, a cardboard sword, and cardboard cakes for each group or, better yet, have the groups make them. The cakes should be non-homogeneous; hence participants may prefer some part.



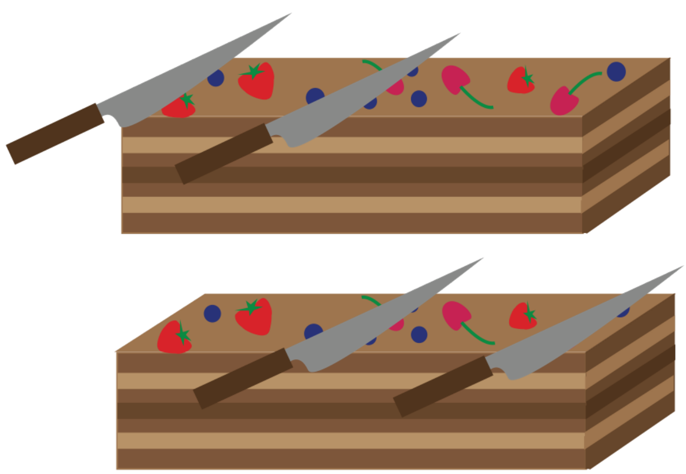
Activity 1 - An envy-free way to share cake between two people

The goal is to split and share a cake between two people so that neither envies the piece received by the other.

- Ask the group if anyone can think of a strategy to accomplish this.
- Propose the strategy: "I cut, you choose" and try it several times with two participants. Suppose Maryam cuts and Caucher chooses. How should Maryam cut, so she will not envy Caucher? Such a division is called "envy-free."



- Discuss: Is it better to be the one who cuts or the one who chooses?
- Since it's better to choose, Maryam proposes to refine the previous strategy. She says to Caucher: *"I will move two knives continuously from left to right over the cake. When you tell me to stop, I will stop and cut the cake along two knives' positions. Then I will choose either the piece between the two knives, or the two end-pieces, and you get the other share."*
- Discuss: How should Maryam hold her knives? When should Caucher say "stop"?



- If the group of participants does not find the strategy, explain it: Maryam should hold her two knives to always be happy with any of the two shares. Caucher should say "Stop!" when he is equally happy with the two shares.

- But why are we sure that there is at least one time when Caucher will be equally happy with the two shares? If the participants find no explanation propose and discuss the one below.

- Here is the explanation of why there is at least one time when Caucher will be equally happy with the two shares. In the beginning, one knife is on the left edge of the cake, and the other is dividing the cake into two portions. Suppose that Caucher prefers the right piece (which is the share “outside the knives”). At the other end, when one knife is on the right edge, the same preferred piece is now between the knives. Then, there has been a continuous movement of the knives in which, at the beginning Caucher preferred the share outside the knives, while at the end, he preferred the share between the knives. Hence, in between, he has passed a position where the two shares are equally likely to him (this is an application of the Intermediate Value Theorem).

- We call such a division “equitable” because both Maryam and Caucher receive a piece that has equal value to them. An equitable division is envy-free, but the inverse may not be the case.

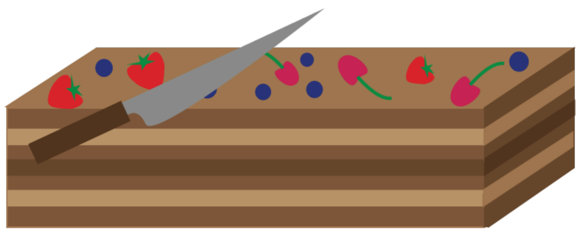
Activity 2 - A proportional division of cake between n people

Here the goal is that each participant receives a portion that they value as at least 1/n of the full cake.

- Ask if anyone wants to propose a method.

- Here is one method: The knife is now in the hand of a mediator, who moves it from left to right. As soon as one participant says “stop!” the mediator stops and cuts the cake along the knife’s position. The participant who said “Stop!” receives the piece to the left of the knife.

- Then the mediator starts again to move the knife on the remaining piece of cake until a second participant says “Stop!”. That participant receives the new piece to the left of the knife, and so on. The last participant (who never said “Stop!”) gets the last remaining piece.



- Try this method several times with the whole group: the mediator should move the knife very, very, slowly so that participants have time to decide their strategy. Do some participants feel that they have received a piece that is less than their share of 1/n of the whole cake? If so, how can they modify their strategy?

- Here is a strategy for a proportional division: each participant says “Stop!” whenever they find that the value to the left of the knife is worth to them at least 1/n of the full cake’s total value. Explain why this strategy ensures that each participant will receive a piece that they value as at least 1/n of the full cake.

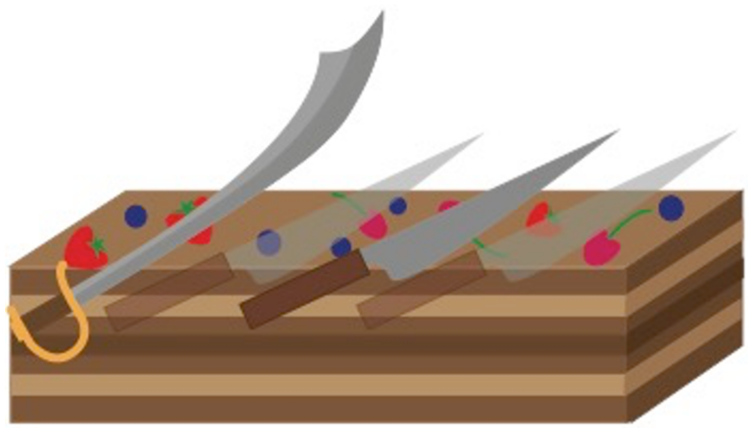
- Note that this strategy is not envy-free. Why not?

Activity 3 - An envy-free division of cake between 3 people

The goal now is to divide the cake so that no participant envies the pieces of cake received by the other two.

- Ask if anyone wants to propose a strategy.

- Here is one strategy: The mediator holds a saber and moves it from left to right. At the same time, each participant moves a knife on the right side of the saber. As soon as one participant says “Stop!” everyone stops moving their knife. The cake is cut in two places: at the positions of the saber and the middle knife. The participant who said “Stop!” receives the piece to the left of the saber. Of the two remaining participants, the one whose knife is the further right gets the right piece, and the remaining participant receives the middle piece.



- Try this method several times with different members taking the role of mediator. The mediator should move the saber very slowly so that participants have time to decide their strategy. Do some participants envy the share of another participant? Does anyone want to propose a strategy for how each participant should move their knife?

- Here is a strategy for an envy-free division:
    - Each participant holds their knife so that the two pieces delimited by it, to the right of the saber, are of equal value to them.
    - Let’s call “participant 1” the one whose knife is the farthest to the left, “participant 2” the one whose knife is in the middle, and “participant 3” the one whose knife is the farthest to the right.
    - Participant 1 says “Stop!” whenever they find that the left piece has the same value to them as the middle piece.
    - Participant 2 says “Stop!” when the three pieces have the same value to them.
    - Participant 3 says “Stop!” whenever they find that the value of the left piece is the same to them as the value of the right piece.

- Discuss why this strategy is envy-free. Take each participant's point of view and explain why no participant envies the others' share.

Activity 4 - An envy-free division of chores between 3 people

If we lay chores in a grid, we can divide them like a cake. In this case, each participant will want to get a portion that is as small as possible. In this case, our goal is that no participant envies any of the others' share of the chores.

- This problem is analogous to Activity 3, and we can derive a strategy from it. Does anyone have an idea?

- Here is one strategy. The mediator now holds a saber and moves it from left to right. At the same time, each person moves a knife to the right of the saber. As soon as one participant says "Stop!" everyone stops moving their knife. The grid is cut in two places: at the positions of the saber and the middle knife. The participant who said "Stop!" receives the rightmost piece. Of the two remaining participants, the one whose knife is farthest to the right gets the middle piece, and the last one gets the left piece.

Cleaning the windows	**Doing laundry**	**Cooking**	**Ironing**
Cleaning the floors	**Washing the dishes**	**Taking out the trash**	**Feeding the cat**
Vacuuming	**Watering the plants**	**Setting the table**	**Shopping for groceries**
Cleaning the bathroom	**Mowing the lawn**	**Sweeping**	**Recycling**

- Try this method several times with different members taking the role of mediator. The mediator should move the sabor very slowly so that participants have time to decide their strategy. Do some participants envy the share of another participant? Does anyone want to propose a strategy for how each participant should move their knife?

- Here is a strategy for an envy-free division of chores:
    - All participants hold their knife so that the leftmost piece has the same value, to them, as the piece between the saber and their own knife.
    - Let's call "participant 1" the one whose knife is farthest to the left, "participant 2" the one whose knife is in the middle, and "participant 3" the one whose knife is farthest to the right.
    - Participant 1 says "Stop!" whenever they find that the value of the left piece has the same value as the right piece.
    - Participant 2 says "Stop!" when the three pieces have the same value to them.
    - Participant 3 says "Stop!" whenever they find that the middle piece's value is the same as the right piece's.

- Discuss why this strategy is envy-free. Take each participant's point of view and explain why no participant envies the others' share.

Create and Share!
Share pictures and videos of the activity or strategies proposed by the group, using the hashtag **#idm314**.

Go deeper with some videos on the subject:

- See a different method to share a cake between three people, explained by Hannah Fry in a Numberphile video.
- Math Encounters - Fair Division: How to Cut Cakes (and other things) Fairly. A talk about fair division by Professor Francis Su in the MoMath museum.

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